# Heat and Moisture Diffusion in Magnetic Tape Packs

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**ABSTRACT**

The purpose of this paper is to present theory and experiments on heat and moisture diffusion in magnetic tape packs. The theory includes anisotropy effects. The same fundamental mathematical theory is applied to heat and moisture diffusion. The tape packs have been submitted to a variety of thermal and hygroscopic conditions. By fitting experimental curves, a heat diffusivity coefficient of $1.8 \times 10^{-7}$ m²/s was obtained for the radial direction. Separate experiments showed that the heat diffusivity in the axial direction is about twice that value.

For the radial moisture diffusivity coefficient, a literature value of $4.0 \times 10^{-13}$ m²/s is used. Fitting experimental curves then shows that the axial diffusion is about 300 to 500 times faster than the radial diffusion which indicates that most of the moisture enters the pack through the top and bottom edges.

Finally, it is shown that the experimental pack boundaries are not at a fixed temperature. Heat diffusion experiments in a stirred water bath best approximate the constant temperature requirement and theoretical fits to those experimental curves have been most satisfactory.

## INTRODUCTION

Magnetic tape is an important information storage medium for both short and long term uses. The temperature and moisture content of magnetic tape media are known to affect the physical, chemical, and magnetic properties of the tape. Thus, the temperature and moisture content are important when magnetic tape is being recorded or played in a tape facility, stored in an archive, or transported. Magnetic tape is typically used in pack form on metal or plastic reels and is commonly encased in cassette form. It is useful to know and understand the amount of time it takes for magnetic tape in reel or cassette form to respond to temperature or moisture changes in the environment. This information is needed in the development of tape handling procedures and in evaluating the time-dependent effects of temperature and moisture on magnetic tape.

Prior research in the area of thermal and hygroscopic properties of polymer films has been conducted by several authors. However, the measurement and theoretical treatment of these properties in the specialized case of magnetic tape media in pack form have only been considered in a few instances [1,2]. Cases of thermal and hygroscopic changes in tape packs are reported by Cuddihy [3,4]. It is the goal of this paper to extend the theoretical and experimental treatment of thermal and moisture changes in magnetic tape packs. This is accomplished by extending the previously published theory to describe a hollow cylindrical shape that resembles a tape pack. The developed models are then compared to experimental data obtained from thermal and hygroscopic measurements on tape packs responding to changes in the ambient environment.

## THEORY

The theory will be developed for tape pack temperature changes first. This will be followed by a treatment of moisture diffusion.

### Thermal Theory:

The fundamental heat equation with $\theta$ the absolute temperature, $t$ the time and $\kappa$ the diffusivity coefficient is given as [5]:

$$\nabla \cdot (\kappa \nabla \Theta) = \frac{\partial \Theta}{\partial t}. \qquad (1)$$

In Figure 1, a cylindrical coordinate system is defined for the tape pack under consideration. Since the pack is a layered structure, anisotropic heat flow may be expected. Anisotropy effects may lead to a flow not perpendicular to a plane of constant temperature [5, p.38], resulting in non-zero off-diagonal terms in the matrix for $\kappa$. However, since the principle axes for the internal layered structure coincide with the cylindrical $r$, $\phi$, $z$ axes and since the tape pack boundaries are at the same constant temperature, a flow perpendicular to a plane of equal temperature is naturally expected. Therefore, the matrix for $\kappa$ is diagonal [5, p.41] with constant diagonal elements $\kappa_r$ , $\kappa_\phi$ and $\kappa_z$. The $\kappa_r$, $\kappa_\phi$ and $\kappa_z$ are the thermal diffusion coefficients in the radial, tangential, and axial directions, respectively.

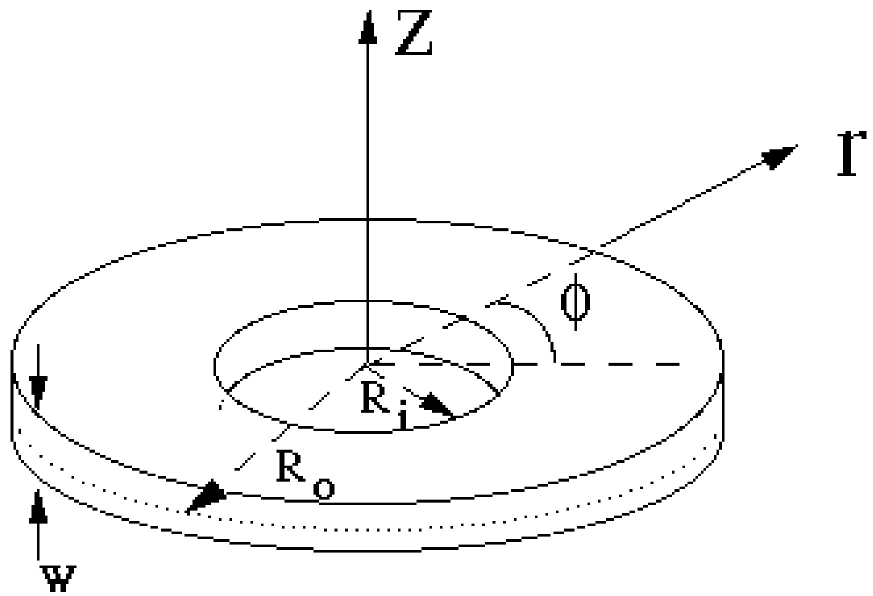


**Figure 1: Tape pack geometry in cylindrical coordinates.**

For symmetry reasons, both the temperature and heat flow will not depend on $\phi$. In cylindrical coordinates with no $\phi$-dependence equation (1) becomes:

$$\frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2} + \frac{\kappa_z}{\kappa_r}\frac{\partial^2 \Theta}{\partial z^2} = \frac{1}{\kappa_r}\frac{\partial \Theta}{\partial t}. \qquad (2)$$

Experimentally, the cylindrical tape pack is initially conditioned to a uniform temperature $T_i$. The pack is then placed in an environment at a constant ambient temperature of $T_a$ at time $t = 0$. A new function $u(r,z, t)$ can now be defined, equal to:

$$u(r, z, t) = \Theta(r, z, t) - T_a, \qquad (3)$$

so that the initial condition inside the tape pack volume is:

$$u(r, z, 0) = T_i - T_a \quad R_i \le r \le R_o \quad -\frac{w}{2} \le z \le \frac{w}{2}, \qquad (4)$$

with boundary conditions

$$u\left(r, \pm\frac{w}{2}, t\right) = 0 \qquad t > 0 \qquad (5)$$

$$u(R_i \text{ or } R_o, z, t) = 0 \qquad t > 0. \qquad (6)$$

The function u(r,z,t) is solved using the separation of variables technique where u(r,z,t) is defined as a product of three functions R(r), Z(z) and T(t). The result of solving for the 3 independent functions is:

$$R(r) = J_0(\mu_m r) Y_0(\mu_m R_o) - J_0(\mu_m R_o) Y_0(\mu_m r) \qquad (7)$$

$$Z(z) = \cos(\nu_n z) \qquad (8)$$

$$T(t) = e^{-\lambda_{mn}^2 \kappa_r t}. \qquad (9)$$

Here $J_0$ is the zero-order Bessel function of the first kind and $Y_0$ the zero-order Bessel function of the second kind (Neumann function). The constant $\mu_m$ ($m$ = 1, 2, 3, . . .) is obtained by applying the boundary condition (6) and finding the roots of equation (7) at $r = R_i$:

$$J_0(\mu_m R_i) Y_0(\mu_m R_o) - J_0(\mu_m R_o) Y_0(\mu_m R_i) = 0. \qquad (10)$$

The constant $\nu_n$ ($n$ = 1, 2, 3, . . .) is obtained by applying the boundary condition (5) and is equal to:

$$\nu_n = \frac{(2n - 1)\pi}{w} \qquad (11)$$

and also

$$\lambda_{mn}^2 = \mu_m^2 + r_c^2 \nu_n^2 \qquad (12)$$

with the anisotropy ratio equal to $\kappa_z/\kappa_r = r_c^2$ in (12).

The total solution for $u(r,z,t)$ is now given by any linear combination of the product $R(r)Z(z)T(t)$ subject to the initial condition (4) and boundary conditions (5) and (6).

$$u(r, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} R(r) \cos(\nu_n z) e^{-\lambda_{mn}^2 \kappa r t}. \quad (13)$$

The constant $a_{mn}$ is determined from initial condition (4) by solving for the Fourier series coefficients and is given by (see, e.g., [5, p.207]):

$$a_{mn} = (T_a - T_i) \frac{4(-1)^n}{(2n - 1)} \frac{J_0(\mu_m R_i)}{J_0(\mu_m R_i) + J_0(\mu_m R_o)}. \quad (14)$$

Let $R_m = J_0(\mu_m R_i)/(J_0(\mu_m R_i) + J_0(\mu_m R_0))$. Using (3), the final solution for $\theta(r, z, t)$ now becomes:

$$\Theta(r, z, t) = T_a + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (T_a - T_i) \frac{4(-1)^n}{(2n - 1)} \cdot R_m R(r) \cos(\nu_n z) e^{-\lambda_{mn}^2 \kappa r t} \quad (15)$$

Equation (15) has been used to fit the experimental data. The results are presented in following sections.

**Hygroscopic Theory:**

In the collection of the experimental data to follow and in actual use of the magnetic tape, the concentration of moisture in the tape pack at equilibrium is roughly proportional to the relative humidity of the surrounding air. A simple calculation based on the data provided by Cuddihy [3], reveals that the tape moisture concentration in equilibrium is about 3 orders of magnitude higher than that of the surrounding ambient air. The same reference also indicates that the water mass in a tape pack is roughly proportional to the relative humidity of the surrounding air. In addition, Henry's law has been shown to be obeyed in polyethylene terephthalate (PET) [6] which, by volume, is the major component of magnetic tape. It is therefore assumed throughout this analysis that the boundary conditions for moisture in the tape pack can be controlled by controlling the relative humidity of the surrounding air and that the equilibrium concentration of moisture at the surface of the tape pack is always linearly proportional to the relative humidity of the surrounding air.

It is assumed that the movement of water molecules inside the tape pack is governed by diffusion. In addition, it is assumed that the moisture diffusivity coefficient $D$ is independent of the water molecule concentration $\Psi$. Thus, analogous to (1), the fundamental moisture diffusion equation is given as [7]:

$$\nabla \cdot (D \nabla \Psi) = \frac{\partial \Psi}{\partial t}. \quad (16)$$

Experimentally, the cylindrical tape pack is initially conditioned to a uniform water concentration $C_i$ at a certain relative humidity and is then at time t = 0 placed in an environment with a different relative humidity which will result in a water concentration $C_a$ for the entire tape pack at $t \to \infty$. A comparison of equation (16) to equation (1) shows that the moisture diffusion solution is identical to the temperature diffusion solution with the substitution of $C$ for $T$ and $D$ for $\kappa$. Accordingly, the solution of the moisture concentration $\Psi$ yields:

$$\Psi(r, z, t) = C_a + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (C_a - C_i) \frac{4(-1)^n}{(2n - 1)} \cdot R_m R(r) \cos(\nu_n z) e^{-\lambda_{mn}^2 D_r t} \qquad (17)$$

In the case of moisture diffusion, the readily measurable quantities are the environmental relative humidity and the mass of the tape pack. The total moisture mass $M(t)$ is obtained by integrating the moisture concentration $\Psi(r, z, \phi, t)$ over the pack volume:

$$M(t) = \int_0^{2\pi} \int_{-w/2}^{w/2} \int_{R_i}^{R_o} \Psi(r, z, \phi, t)\, dr\, dz\, r d\phi. \qquad (18)$$

Completing the above integration with the initial moisture mass $M_i$ equal to

$$M_i = w\pi(R_o^2 - R_i^2) C_i \qquad (19)$$

yields:

$$M(t) = M_a + \frac{M_a - M_i}{w\pi(R_o^2 - R_i^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(-1)^n}{(2n - 1)} \cdot e^{-\lambda_{mn}^2 D_r t} R_m \int_0^{2\pi} \int_{-w/2}^{w/2} \int_{R_i}^{R_o} R(r) r dr \cos(\nu_n z)\, dz d\phi, \qquad (20)$$

where, analogous to (19), the equilibrium moisture mass $M_a$ is given by

$$M_a = w\pi(R_o^2 - R_i^2) C_a. \qquad (21)$$

Evaluation of the integrals in (20) in combination with (14) provides:

$$M(t) = M_a + \frac{32(M_i - M_a)}{\pi^2(R_o^2 - R_i^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \frac{J_0(\mu_m R_i) - J_0(\mu_m R_o)}{\mu_m^2[J_0(\mu_m R_i) + J_0(\mu_m R_o)]} e^{-\lambda_{mn}^2 D_r t} \qquad (22)$$

## EXPERIMENTS

### Thermal Experimental Procedures:

The results reported here were obtained using 13 um thick by 19 mm wide tape and 24.4 um thick by 25.4 mm wide tape. Both formats are commercially available in plastic cassettes. The cassettes used were D1 format cassettes in sizes suitable for 19 mm and 25.4 mm wide tapes. Experiments were performed both on individual tape reels and on tape reels encased in a cassette. Individual tape reel experiments used a tape pack wound on a reel which had the plastic flanges removed. For experiments performed inside of the cassette, a tape pack on a reel with plastic flanges was used.

Several experiments were performed to determine the influence that the ambient environment had on tape pack heating behavior. The environments investigated included a stirred water bath, a circulating air oven, a still air oven, and a cassette in a still air oven. For bare tape reel experiments, the bare reel was suspended in the fluid of interest. For cassette experiments, the cassette was supported so that air could circulate around the entire cassette.

Thermocouples were used to monitor local temperatures within the tape pack and ambient environment. Type T thermocouples made from 36 gauge wire were used. Thermocouple extension cables made of 24 gauge wire were used to connect the fine gauge wire to an A/D converter board interfaced with a computer that recorded temperature readings. The placement of thermocouples used for tests on tape reels inside a cassette is shown in Figure 2.

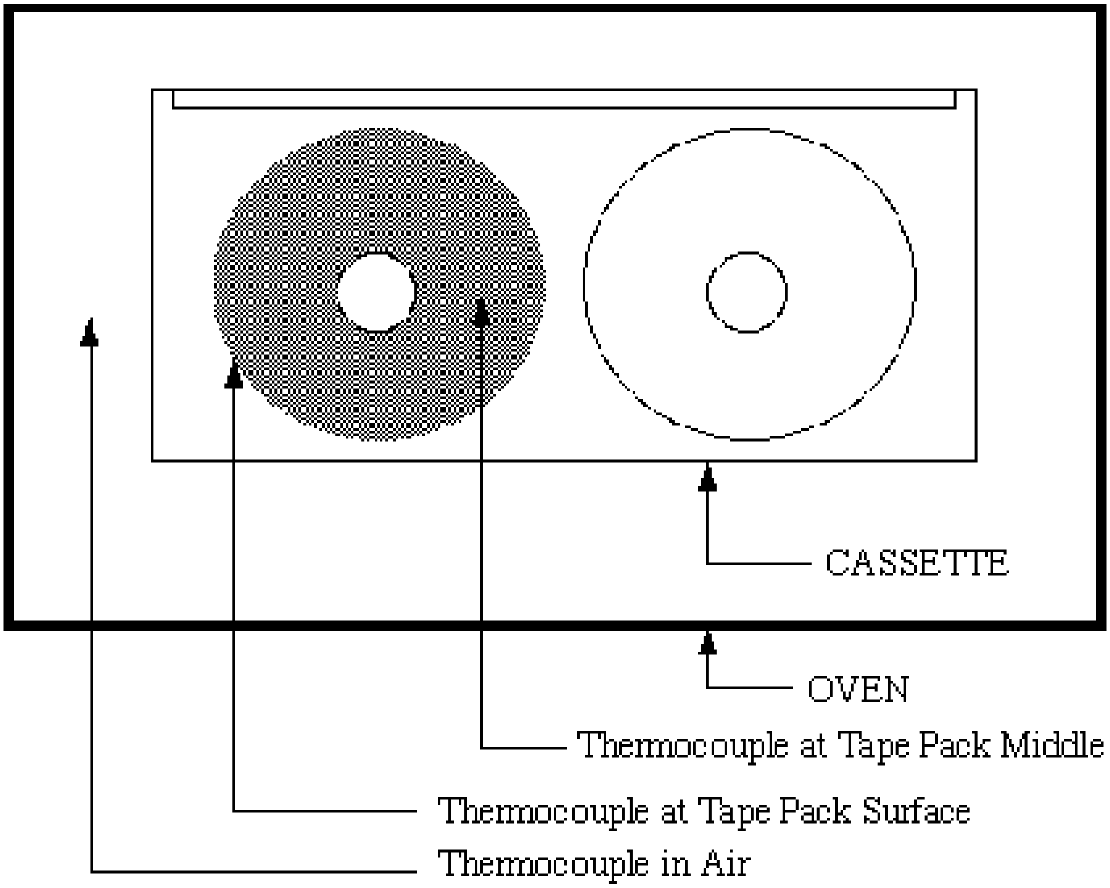


**Figure 2: Arrangement of thermocouples in the cassette and tape pack.**

For cassette experiments, a 0.040 inch diameter hole was drilled into the plastic flange of the tape reel at a position 0.75 inches from the outside of the reel hub. The thermocouple was placed in the "middle" of the tape pack as tape was wound onto the reel. The "middle" of the tape pack refers to a position halfway between the outer and inner tape windings and halfway between the edges of the tape. The reel hub diameter is 1.75 inches and the outer tape pack diameter is approximately 4.5 inches. A thermocouple was attached to the outside of the tape pack using a small piece of electrical tape. The tape reel was then placed inside of the cassette housing in the location which it would normally occupy.

For individual tape reel experiments, the tape reel was prepared in the same manner as described above, except that the tape was spooled onto a reel hub with the plastic flanges removed. The locations of the thermocouples were the same for each experiment. For individual tape reel experiments, the tape reel was not encased in a cassette housing.

Tape reels and cassettes were allowed time to equilibrate to room temperature for a minimum of 18 hours before beginning the experiments. An experiment began when a tape reel or cassette was placed in a warm environment maintained at ca. 60°C. The thermocouple temperatures were recorded until the tape reel reached an equilibrium temperature of ca. 60°C as indicated by the thermocouple located at the "middle" of the tape pack.

Warming environments were selected which differed in their fundamental heat transfer characteristics.

Ranked in an order of highest to lowest heat transfer coefficients, the environments were (1) a stirred water bath, (2) a circulating air (forced air) oven, and (3) a calm air chamber. The 60°C water bath consisted of 3 liters of water in a stainless steel beaker located in a forced air oven at 60°C. The water bath was rapidly stirred with a magnetic stirrer. The calm air chamber consisted of a stainless steel box (180 x 240 x 360 mm) located in a forced air oven at 60°C.

**Thermal Experimental Data:**

The experimental data for thermal equilibration of 25.4 mm wide tape packs is shown in Figures 3 through 6. The curves differ in the environment used for warming. Data in Figure 3 is for a tape reel immersed in a stirred water bath at 60°C. Data in Figure 4 is for a tape reel placed in a circulated air oven 60°C. Data in Figure 5 is for a tape reel placed in calm (stagnant) air at 60°C. Data in Figure 6 is for a tape reel in a cassette placed in calm air at 60°C.

**Discussion of Thermal Data:**

Equation (15) was derived assuming the surface temperature of the tape pack immediately changes and becomes equal to the temperature of the new environment when a tape pack is relocated. Whether or not this condition wt is for this thermal boundary layer (e.g., a fluid with a higher thermal conductivity), or the smaller the thickness of this layer is (e.g., a fluid with greater circulation), the more quickly the tape pack surface temperature will approach that of the ambient environment.

Reviewing the experimental data, the discrepancy between a theoretical constant surface temperature boundary condition and the actual experimental tape pack surface temperatures increases as the media surrounding the tape pack is changed from stirred water to circulated air to calm air and finally to cassette shell in calm air. The experiments are ordered from highest to lowest ambient environment heat transfer coefficient.

The results for the stirred water bath experiment, with the largest fluid heat transfer coefficient, most closely match the constant surface temperature boundary conditions expressed in equations (5) and (6). The surface temperature of the tape pack (see Figure 3) rapidly rises to within a few degrees of the bath temperature. After 30 seconds, the surface temperature of the tape pack is relatively constant. Thus, one might expect equation (15) to satisfactorily model the experimental response of the middle tape pack temperature.

The experimental middle tape pack temperature is plotted together with the theoretical fit obtained using equation (15) in Figure 3. The agreement between data and theory is excellent to the extent that the two curves cannot be discerned from one another.

In a separate independent experiment measuring temperature differences across the tape pack in both the axial and radial directions, a value of 2 for the anisotropy ratio $\kappa_z/\kappa_r$ was obtained. This anisotropy value was fixed during the fitting procedure. After the fitting procedure, we obtained a value for the thermal diffusivity for the pack in the radial direction $\kappa_r$ of $1.8 \times 10^{-7}$ m²/s (see Fig. 3). For comparison, reference [4] uses known values for the thermal conductivity, heat capacity and density of PET, resulting in a thermal diffusivity value of $0.95 \times 10^{-7}$ m²/s.

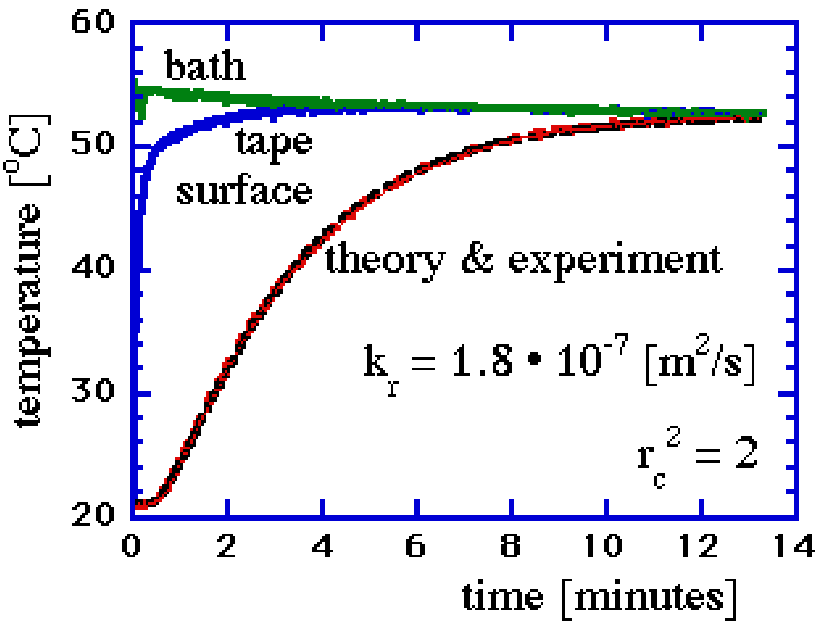


**Figure 3: Thermal changes in a 25.4 mm wide bare tape pack in a stirred water bath**

The theoretical fit to the experimental data is excellent, although the experimental tape pack surface temperature was not quite constant as the theoretical boundary condition requires. This results from the relative insensitivity of the initial portion of the curve to tape pack surface temperature variations. For this particular experiment, a ±2° C change in the tape pack surface temperature resulted in a difference of only ±0.2° C in the theoretical curve for times less than 1.5 minutes. An exact fit at long times was assured by chosing the tape pack surface temperature $T_a$ equal to the final bath temperature.

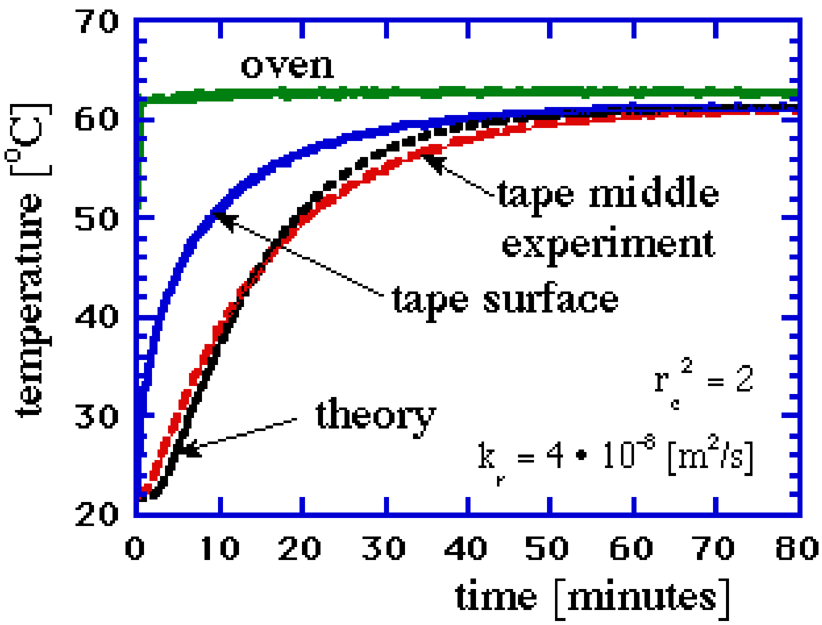


**Figure 4: Thermal changes in a 25.4 mm wide bare tape pack in circulated air**

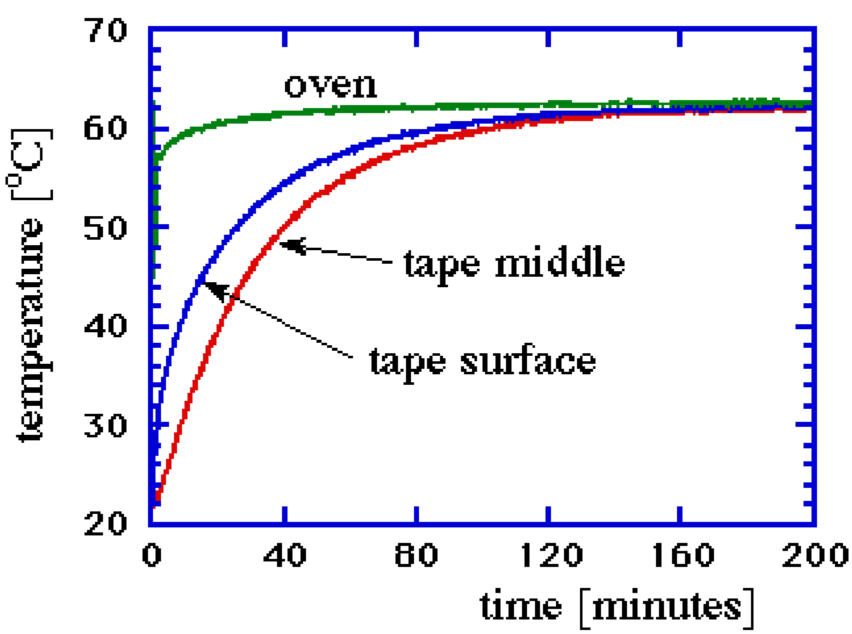


**Figure 5: Thermal changes in a 25.4 mm wide bare tape pack in calm air**

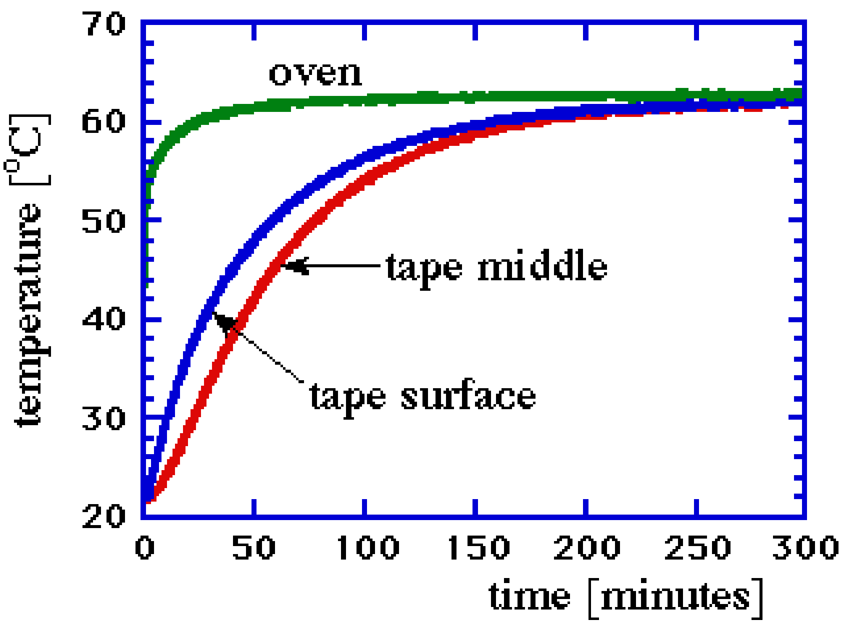


**Figure 6: Thermal changes in a 25.4 mm wide tape pack in a cassette in calm air**

Results for the experiment using a circulated air environment are shown in Figure 4. In this instance, the tape pack surface temperature does not follow the oven temperature and more closely follows the tape pack middle temperature. Conditions of a constant surface temperature are not attained in this experiment. In Figure 4, a theoretical fit to the tape pack middle temperature is plotted along with the experimental data. A value for the thermal diffusivity for the pack in the radial direction, $\kappa_r$, of $4.0 \times 10^{-8}$ m²/s and an anisotropy ratio, $\kappa_z/\kappa_r$, of 2 is obtained by applying the theoretical model to the experimental data. The agreement between data and theory is not good in this case. The discrepancy between theory and experiment arises because the boundary conditions are not met by the experiment. The low thermal conductivity of air results in a large temperature gradient in the air immediately surrounding the tape pack. This thermal boundary layer results in the lower surface temperatures observed.

In still air, the heat transfer coefficient is lower than that for circulating air, resulting in a larger boundary layer thickness. The effect of this larger boundary layer can be seen in Figure 5, where a slower overall heating of the tape pack is observed relative to Figure 4.

In use, tape packs are often encased in plastic cassettes for convenience and protection. Figure 6 shows experimental results for the heating rate of a tape pack encased in a plastic cassette. By encasing a tape pack in a cassette, the time required to reach thermal equilibrium is approximately doubled as compared to that for an individual tape reel (Figure 5). The tape pack surface temperature data indicates the inappropriateness of constant surface temperature boundary conditions. Thus, equation (15) is not an appropriate model for the heating and cooling of tape packs in cassettes.

**Hygroscopic Experimental Procedures:**

Changes in the moisture content of cassettes were measured by the mass change of the cassettes. The mass changes were measured using models PM1200 and PM4000 Mettler precision balances.

One cassette was placed in an environmentally controlled room at 72°C and 80% relative humidity and a second similar cassette was placed in a room conditioned to 72°C and 20% relative humidity. The mass of the cassettes was recorded during this conditioning process so that the cassettes were known to be at a moisture content in equilibrium with the room before the start of the experiment. To begin the experiment, the conditioned cassettes were exchanged--the sample conditioned at 20% relative humidity was placed in the 80% relative humidity room and vice versa. The mass of each sample was recorded once every hour as the samples equilibrated to the humidity of their new environments.

**Hygroscopic Experimental Data:**

A comparison of the hygroscopic behavior of tape in a bare pack wound on a metal hub and the same type of tape in a cassette is shown in Figure 7. The bare pack and cassette have different total weights so that the mass change of each sample is normalized by the weight change observed at long time so that a comparison between the two samples can be made.

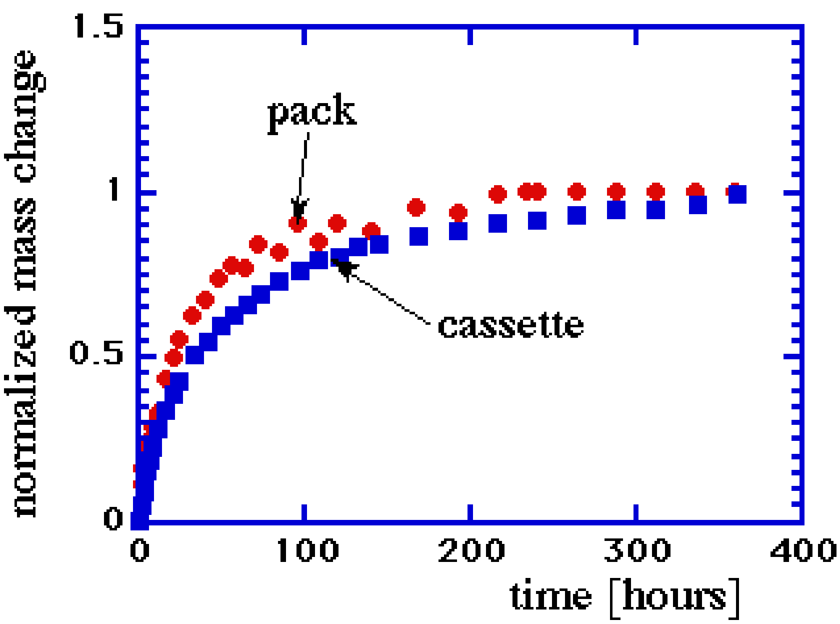


**Figure 7: Moisture diffusion profiles for a bare tape pack and for a tape pack in a cassette**

Two similar samples of 19 mm wide tape enclosed in cassettes were weighed as a function of time with time equal to zero indicating the first exposure to a new environment. The experimental results compared to the theoretical result in equation (22) are shown in Figure 8.

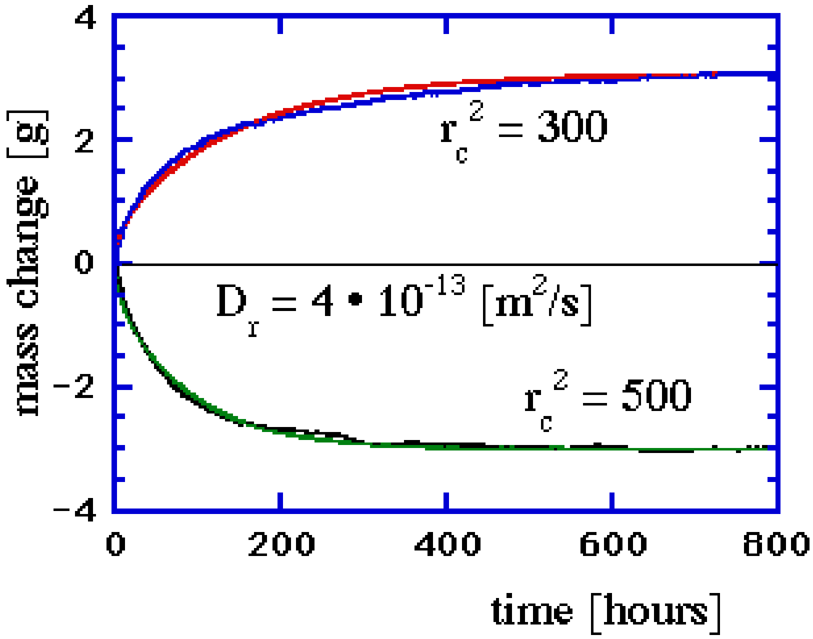


**Figure 8: Moisture diffusion induced changes in a 19 mm wide tape cassette with the anisotropy constants determined by fitting the theory to the data.**

**Discussion of Hygroscopic Data:**

Figure 7 indicates that the behavior of moisture diffusion in a bare tape pack is very close to the behavior of the tape in an enclosed cassette. Figure 8 indicates that equation (22) describes the moisture change in cassette tape packs contained in cassettes reasonably well. The physical reason for the similarity of the bare pack and cassette data is that moisture diffuses through the cassette and the air surrounding the tape pack at a much faster rate than moisture diffuses into the tape pack. Thus, there is not a significant diffusion boundary layer surrounding the tape pack. This results in the boundary conditions in equations (5) and (6) being satisfied in the cassette hygroscopic experimental case; unlike the thermal case.

In Figure 8 a value of $4.0\text{x}10^{-13}$ m²/s is used for $D_r$, the moisture diffusivity in the radial direction, and the anisotropy ratio $r_c^2$ is varied to produce good fits to the experimental data for the theoretical curves. The value for $D_r$ was selected from the literature available for PET in reference [6]. This value was compared to the values obtained experimentally by performing moisture vapor transmission measurements on single layers of magnetic tape and PET. These types of measurements are detailed in ASTM E96-80 entitled "Standard Test Methods for Water Vapor Transmission of Materials." Values obtained for PET film and for magnetic tape under the same conditions are 1.1 and 0.92 respectively. The values obtained for $r_c^2$ are considered reasonable when the spacing between adjacent tape layers is taken into account. The $r_c^2$ values on the curves appear to differ depending upon whether the tape pack weight is increasing or decreasing. We have observed the same tendency in the two independent experiments that we performed. This behavior is physically reasonable as the tape pack swells when moisture is gained thus reducing the space between the tape layers and the amount of moisture diffusing through the top and bottom of the pack. The opposite is true if the pack is losing moisture. This implies that the exact value for $r_c^2$ will depend upon how the pack is wound since this influences the amount of space between adjacent tape layers.

## CONCLUSIONS

We have shown that the change in temperature and moisture content in magnetic tape packs can be described by the heat diffusion equation for a hollow cylinder. The simple theoretical boundary conditions for describing the temperature changes in a magnetic tape pack are best satisfied when the surrounding medium is stirred water. For the cases of circulated and calm air as the heat environment, the simple theoretical boundary condition is not accurate. The thermal and moisture diffusivity coefficients of tape packs have been estimated by fitting theory to experiment using commercial tape and cassettes. In both thermal and moisture diffusion cases the tape pack is shown to be anisotropic and estimates for this anisotropy have been obtained.

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